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Low-energy-theorem approach to one-particle singularity in QED$_{2+1}$

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Abstract: We evaluate the spectral function of scalar and spinor propagator in three dimensional quantum electrodynamics with the use of Ward-Identity for soft-photon emission vertex. Exponentation of one-photon matrix element yields a full propagator in position space. It has asimple form as free propagator multiplied by quantum correction. But this is not an integrable function so that analysis in momentum space is not easy. Term by term integral converges and they have a logarithmic singularity associated with renormalized mass in perturbation theory. Renormalization constant vanishes for weak coupling,which suggests confinement of charged particle. There exsists a critical coupling constant below which bare mass vanishes and the vacuum expectation value of pair condensation is finite.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Confinement.

Contents

1. Introduction

In three dimensional gauge theory infrared divergences is severer than that in four dimension. In 1981, Jackiw first demonstrated it in the massless fermionic QED_{2+1} and after that other authors introduced the infrared counter terms to renormalize the infrared divergences [1, 2]. There had been an attempts to solve Dyson-Shwnger equations to examine the dynamical mass generation or chiral symmetry breakings in this model to improve low-energy behaviour [3, 4]. Another important feature of this model is that it allows parity violating Chern-Simons term in the Lagrangean without violating gauge invariance [5]. But it has not yet been clear the quantum effects of Chern-Simons term. During the same period infrared behaviour of the propagator in the presence of dynamical mass has been analysed to search the physical cut associated with massless photon [6, 7]. To maintain gauge covariance Delbourgo and Jackiw addded the vertex correction with Ward-Identities for soft particles in the determination of the infrared behaviour of the propagator with bare mass [9, 10]. Their works are fundamental and important to understand the effects of soft-photons in comparison with perturbation theory. Of course their works includes non-perurbative effects. In 1992 Delbourgo applied his method (gauge thecnique) to QED_{2+1} and shown that massless loop correction to photon soften infrared divergences as T. Appelquist et.al. [3, 6]. In reference [9] infrared behaviour of the scalar propagator in the presence of massless fields (photon, graviton) in four dimension was determined by solving the spectral function. In this work we apply the same method in three dimension to determine the infrared behaviour of the propagator. In section 1 we analyse scalar QED_3 and evaluate the scalar propagator in position space. In this case an approximaiton is made to choose the one-photon,one-meson intermediate state to derive the matrix element $\langle\Omega|\phi(x)|n\rangle$ for the spectral function. Most singular infrared part is assumed to be the contribution of soft photon emitted from external lines. Therefore we introduce photon mass to avioid infrared divergences. Exponentiation of the lowest-order spectral function yields the full propagator in position space. Its has a simple form as free propagator (with renormalized mass) multiplied by quantum correction. However this function is not integrable and it is difficult to make a fourier transform of it. In the pertuebative analysises the propagator have linear and logarithmic infrared divergences near on-shell. We mention the gauge transformation property and see our solution satisfies Landau-Kharatonikov transformation. As far as the renormalization constant is concerned we evaluate it by position space propagator. The result is $Z = 0$ for weak coupling,which is a signpost of confining phase in our approximation. In section 2 we study QED$_3$ with spinor and see the spectral function for fermion is the same in section 1. In this case there is a interesting possibility of pair condensation. Evaluating the vacuume expectation value $\langle\overline{\psi}\psi\rangle$ and we find that there exists a critical coupling constant $(e^2/(8\pi m) = 1)$ below which the bare mass vanishes and $\langle\overline{\psi}\psi\rangle$ remains finite.

2. Scalar QED

First we assume the Kallen-Lehmann spectral representation of the propagator for massive boson [1]:

$$\Delta(p^2) = \int d^3x \exp(ip\cdot x) \langle\Omega|T\phi(x)\phi^+(0)|\Omega\rangle\,, \tag{2.1}$$

$$\Delta(p^2) = \int \frac{\sigma(s)ds}{p^2 - s + i\epsilon}\,, \tag{2.2}$$

$$\begin{aligned} \sigma(p^2) &= (2\pi)^2 \sum_N \delta(p - p_N) \langle\Omega|\phi|N\rangle \langle N|\phi^+|\Omega\rangle\,, \\ |N\rangle &= |r; k_1, \ldots, k_n\rangle, r^2 = m^2\,. \end{aligned} \tag{2.3}$$

The spectral function is formally written as a sum of multi-photon intermideate states:

$$\begin{aligned} \sigma(p^2) = \int \frac{d^2r}{2r_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int \frac{d^3k}{(2\pi)^2} \theta(k_0)\delta^{(3)}(k^2) \sum_\epsilon \right)_n \delta\left(p - r - \sum_{i=1}^n k_i\right) \times \\ \times \langle\Omega|\phi|r_i; k_1, \ldots, k_n\rangle \langle r; k_1, \ldots, k_n|\phi^+|\Omega\rangle\,. \end{aligned} \tag{2.4}$$

Here we use the notation

$$\begin{aligned} (f(k))_0 &= 1 \\ (f(k))_n &= \prod_{i=1}^n f(k_i)\,. \end{aligned} \tag{2.5}$$

To evaluate the contribution of the soft-photons,we consider from the beginnings when only the nth photon is soft. Define the following matrix element

$$T_n(r; k_1, \ldots, k_n) = \langle\Omega|\phi|r; k_1, \ldots, k_n\rangle\,. \tag{2.6}$$

We consider T_n for $k_n^2 \neq 0$,we continue off the photon mass-shell by Lehmann-Symanzik-Zimmermann (LSZ) formula:

$$T_n = \epsilon_n^\mu T_{n\mu} \,, \tag{2.7}$$

$$T_n^\mu = \int d^3x \exp(ik_n \cdot x)\Box_x \langle \Omega | T\phi A_\mu(x) | r; k_1, \ldots, k_n \rangle \tag{2.8}$$

$$= \int d^3x \exp(ik_n \cdot x) \langle \Omega | T\phi j^\mu(x) | r; k_1, \ldots, k_{n-1} \rangle \,, \tag{2.9}$$

where the electromagnetic current is

$$j^\mu(x) = ie\phi^+(x)\overleftrightarrow{\partial_\mu}\phi(x) - 2e^2 A_\mu \phi^+ \phi \,, \tag{2.10}$$

here we assume the second term does not contribute in the infared behaviour of the propagator and the usual commutation relation for the first term

$$\delta(x_0 - y_0)[j_0(x), \phi(y)] = e\phi(x) \,. \tag{2.11}$$

Thus we omitt the second term in the present calculus. From the definition (2.9), T_n^μ is seen to satisfy the Ward-Identity

$$k_{n\mu} T_n^\mu = eT_{n-1}(r; k_1, \ldots, k_{n-1}) \,. \tag{2.12}$$

Here we mention the simple proof:

$$\begin{aligned} k_{n\mu} T_n^\mu &= \int d^3x k_{n\mu} \epsilon_n^\mu \exp(ik_n \cdot x) \langle \Omega | T\phi j^\mu(x) | r; k_1, \ldots, k_{n-1} \rangle \\ &= i \int d^3x \epsilon_n^\mu \exp(ik_n \cdot x) \partial_\mu \langle \Omega | T\phi j^\mu(x) | r; k_1, \ldots, k_{n-1} \rangle \\ &= e\epsilon_n^\mu \langle \Omega | \phi | r; k_1, \ldots, k_{n-1} \rangle \,, \end{aligned} \tag{2.13}$$

provided

$$\partial_\mu T(\phi(y) j^\mu(x)) = \delta(x - y) e\phi(y) \,, \tag{2.14}$$

where e is the charge carried by meson. To get the solution of the Ward-Identity we seek the non-singular contribution of photon for $T_n(r; k_1, \ldots, k_n)$. There is a possible solution which satisfy Ward-Identity [9]

$$T_n^\mu(r; k_1 \ldots k_n) = \frac{e(2r + k_n)^\mu}{2r \cdot k_n + k_n^2} T_{n-1}(r, k_1 \ldots k_{n-1}), r^2 = m^2 \,. \tag{2.15}$$

Here we assume most singular contributions of photons emitted from external lines as in four dimension. These arise from Feynman diagram in which the soft-photon line with momentum k_n^μ and the incoming meson line can be separated from the remainder of the diagram by cutting a single meson line. Detailed dicussion and evaluation of the explicit form of T_n are given in ref. [9]. Of course for $n = 1$

$$T_1 = \frac{e(2r + k) \cdot \epsilon}{(2r \cdot k + k^2)} \,, \tag{2.16}$$

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and we can determine T_n recursively with eq. (2.16). In this case T_n becomes

$$T_n = \left(\frac{e(2r+k)\cdot\epsilon}{2r\cdot k+k^2}\right)_n . \tag{2.17}$$

The one photon matrix element by LSZ is

$$\begin{aligned} T_1 &= \left\langle in|T(\phi_{in}(x), ie\int d^3y\phi^+_{in}\overleftrightarrow{\partial^y_\mu}\phi_{in}(y)A^\mu_{in}(y))|r;k\ in\right\rangle \\ &= ie\int d^3yd^3z\triangle_F(x-y)\overleftrightarrow{\partial^y_\mu}\epsilon^\lambda_\mu(k)\exp(ik\cdot y)\delta^{(3)}(y-z)\exp(ir\cdot z) \\ &= e\frac{(2r+k)_\mu\epsilon^\lambda_\mu(k)}{(r+k)^2-m^2}\exp(i(r+k)\cdot x)\,. \end{aligned} \tag{2.18}$$

From this lowest order matrix element the function in (2.4) is given

$$\begin{aligned} F &= \sum_{\text{one photon}} \langle\Omega|\phi|r;k\rangle\,\langle r;k|\phi^+|\Omega\rangle \\ &= \int\frac{d^3k}{(2\pi)^2}\exp(ik\cdot x)\delta(k^2)\theta(k^0)\left[\frac{e^2(2r+k)^\mu(2r+k)^\nu\Pi_{\mu\nu}}{(2r\cdot k+k^2)^2}\right], \end{aligned} \tag{2.19}$$

and σ is expressed by

$$\sigma(p) = \int\frac{d^3x}{(2\pi)^3}\exp(ip\cdot x)\int\frac{d^2r}{2r^0}\exp(ir\cdot x)\exp(F)\,. \tag{2.20}$$

Here $\Pi_{\mu\nu}$ is a polarization sum we have

$$\Pi_{\mu\nu} = -\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)-d\frac{k_\mu k_\nu}{k^2}\,, \tag{2.21}$$

and the free photon propagator

$$D_0^{\mu\nu} = \frac{1}{k^2+i\epsilon}\left[g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}+d\frac{k^\mu k^\nu}{k^2}\right]. \tag{2.22}$$

We get

$$F = -e^2\int\frac{d^3k}{(2\pi)^2}\exp(ik\cdot x)\theta(k^0)\left[\frac{(2r+k)^2\delta(k^2)}{(2r\cdot k+k^2)^2}+(d-1)\frac{\delta(k^2)}{k^2}\right]. \tag{2.23}$$

It is natural to set $\delta(k^2)/k^2$ equals to $-\delta^{'}(k^2)$. We see the reason in calculating the one loop self energy of meson diagram,and take imaginary part. This expression is the same as the one given above with the interpretation $\delta(k^2)/k^2 = -\delta^{'}(k^2)$. We must introduce a small photon mass μ as an infrared cut off. Therefore (2.23) becomes

$$F = -e^2\int\frac{d^3k}{(2\pi)^2}\theta(k^0)\exp(ik\cdot x)\left[\delta(k^2-\mu^2)\left(\frac{m^2}{(r\cdot k)^2}+\frac{1}{(r\cdot k)}\right)-(d-1)\frac{\partial}{\partial k^2}\delta(k^2-\mu^2)\right]. \tag{2.24}$$

The evaluation of the integral to σ is described in appendix A. The result is

$$\begin{aligned}
\sigma(x) &= \frac{\exp(-mx)}{4\pi x}\exp(F(x,e,\mu,m))\,, \\
\sigma(p) &= F.T.(\sigma(x)) = \int \frac{d^3x}{(2\pi)^3}\exp(ip\cdot x)\sigma(x)\,, \\
\sigma(x) &= \exp(A)\frac{\exp(-(m+B)x)}{4\pi x}(\mu x)^{-Cx+D}\,, \\
A &= \frac{e^2}{8\pi\mu}(d-2)+\frac{\gamma e^2}{8\pi m}\,, \quad B=\frac{e^2}{8\pi}(d+\gamma-2)\,, \quad C=\frac{e^2}{8\pi}\,, \quad D=\frac{e^2}{8\pi m}\,.
\end{aligned} \tag{2.25}$$

By the definition (2.1)–(2.3) $\sigma(x)$ is a full puropagator. Free propagator in three dimension is

$$\sigma_0(x,m+B)=\frac{\exp(-(m+B)x)}{4\pi x}\,, \qquad \sigma_0(p,m+B)=\frac{1}{(m+B)^2+p^2}\,,$$

and its quantum correction is expressed by $\exp(A)(\mu x)^{-Cx+D}$. We find in this expression that $\sigma(x)$ is finite provided

$$\begin{aligned}
|\sigma(x) &= \exp(A)\frac{\exp(-(m+B)x)}{4\pi x}(\mu x)^{-Cx+D}| = \text{finite}\,, \\
0 &\le \int_0^\infty \sigma(x)dx \le M,\ 0<D\,,
\end{aligned} \tag{2.26}$$

and there exists $\sigma(p)$

$$\sigma(p)=\int_0^\infty \frac{\sin(px)}{p}\sigma(x)dx\,. \tag{2.27}$$

In this case we have not a simple pole at the Yennie gauge $d=2$ in which we see the singularity at $p^2=(m+e^2\gamma/8\pi)^2$. The integral is not anayitic and we cannot get the precise expression for $\sigma(p)$. If we expand $(\mu x)^{-Cx+D}$

$$\begin{aligned}
(\mu x)^{-Cx+D} &= 1+(-Cx+D)\ln(\mu x)+\frac{1}{2}(-Cx+D)^2(\ln(\mu x))^2+ \\
&\quad +\frac{1}{6}(-Cx+D)^3(\ln(\mu x))^3+\cdots\,,
\end{aligned} \tag{2.28}$$

term by term integral of $\sigma(p)$ converges. Using the following integrals

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{\sin(px)}{p}\exp(-mx)\ln(\mu x)dx \\
&= -\frac{\gamma}{p^2+m^2}-\frac{\ln((m^2+p^2)/\mu^2)}{2(p^2+m^2)}-\frac{\ln((m-\sqrt{-p^2})/(m+\sqrt{-p^2}))}{p^2+m^2}\,,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
I_2 &= \int_0^\infty \frac{\sin(px)}{p}\exp(-mx)x\ln(\mu x)dx \\
&= \frac{-m}{(p^2+m^2)^2}\left[\ln\left(\frac{(m-\sqrt{-p^2})}{(m+\sqrt{-p^2})}\right)+\ln\left(\frac{(p^2+m^2)}{\mu^2}\right)-2(1-\gamma)\right]\,,
\end{aligned} \tag{2.30}$$

$\sigma(p)$ up to $O(e^2)$ is given

$$\sigma^{(2)}(p) = \left[\frac{1}{m^2+p^2} + \frac{A}{p^2+m^2} - \frac{mB}{2(p^2+m^2)^2} + DI_1 - CI_2\right], \qquad p = \sqrt{-p^2}\,. \quad (2.31)$$

In this case the renormalization constant Z becomes

$$Z = 1 + A + D\left(-\gamma + \frac{1}{2}\ln\left(\frac{p^2+m^2}{\mu^2}\right)\right)_{p^2\to\infty} \to \infty\,. \quad (2.32)$$

In the Yennie gauge there remains a renormalization

$$Z = 1 + \frac{e^2}{16\pi m}\ln\left(\frac{(p^2+m^2)}{\mu^2}\right)_{p^2\infty} \to \infty\,. \quad (2.33)$$

In Mikowski space, $p^2 \to -p^2$,with dicontinuity

$$\begin{aligned}
\frac{1}{x-i\epsilon} &= P.V\frac{1}{x} + i\pi\delta(x) \\
\frac{1}{(x-i\epsilon)^2} &= P.V\frac{1}{x^2} + i\pi\delta^{'}(x)\,, \\
\frac{1}{x-i\epsilon}\ln(x-i\epsilon) &= i\pi\delta(x)\ln(x) + P.V\frac{1}{x}i\pi\,. \quad (2.34)
\end{aligned}$$

we can determine the the strucure near $p^2 = -m^2$ by the imaginary part of $\sigma(p)$. Here we notice the second-order spectral function $\sigma^{(2)}(p)$

$$\begin{aligned}
\sigma^{(2)}(x) &= \frac{\exp(-mx)}{4\pi x}(1 + A - Bx + (D - Cx)\ln(\mu x))\,, \\
\sigma^{(2)}(p) &= \frac{1}{m^2+p^2} + \frac{A-\gamma D}{m^2+p^2} - \frac{2mB}{(m^2+p^2)^2} - D\frac{\ln(\sqrt{(m^2+p^2)/\mu^2})}{(p^2+m^2)} + \\
&\quad + \frac{Cm}{(p^2+m^2)^2}\left[\ln\left(\frac{(p^2+m^2)}{\mu^2}\right) - 2(1-\gamma)\right]. \quad (2.35)
\end{aligned}$$

$\mathrm{Im}\,\sigma^{(2)}(p)$ reads

$$\begin{aligned}
\frac{\mathrm{Im}\,\sigma^{(2)}(p)}{\pi} &= (1 + A - \gamma D)\delta(p^2-m^2) - \frac{e^2}{4\pi}m(d-1)\delta^{'}(p^2-m^2) - \\
&\quad - \frac{D\theta(p^2-m^2)}{m^2-p^2} + \frac{Cm\theta(p^2-m^2)}{(m^2-p^2)^2}\,. \quad (2.36)
\end{aligned}$$

Usually we estimate the second-order spectral function in the Feynman gauge $d = 1$;

$$\frac{\mathrm{Im}\,\sigma^{(2)}(p)}{\pi}_{d=1} = \left[\left(1 - \frac{e^2}{8\pi\mu}\right)\delta(p^2-m^2)\left[\frac{D}{p^2-m^2} + \frac{Cm}{(m^2-p^2)^2}\right]\theta(p^2-m^2)\right]. \quad (2.37)$$

This shows the renormalization constant Z has an linear infrared divergence as $\mu \to 0$. If we add higher order corrections linear infrared divergences cancell each other in the same mechanism in four dimension [15]–[17]. In the Yennie gauge it has cuts

$$\begin{aligned}
\frac{\mathrm{Im}\,\sigma^{(2)}(p)}{\pi}_{d=2} &= \delta(p^2-m^2) - \frac{e^2 m}{4\pi}\delta^{'}(p^2-m^2) + \\
&\quad + \frac{e^2}{8\pi m}\left(\frac{1}{p^2-m^2} + \frac{2m^2}{(p^2-m^2)^2}\right)\theta(p^2-m^2). \quad (2.38)
\end{aligned}$$

To see the singularity structure near $p^2 = -m^{*2}$ it is better to expand aroud $p^2 = -m^{*2}$. We get

$$\sigma^{(2)}(p) = \left[\frac{1}{m^{*2} + p^2} + (\exp(A) - 1) \times (-CI_2(m \to m^*) + DI_1(m \to m^*) \right] . \tag{2.39}$$

In this case the renormalization constant Z_2 :

$$Z = 1 + (\exp(A) - 1) + D \left(-\gamma + \frac{1}{2} \ln \left(\frac{m^{*2} + p^2}{\mu^2} \right) \right) \tag{2.40}$$

is divergent at $p^2 = -m^{*2}$. There is an interesting contribution from $\exp(F_2)$

$$\begin{aligned} \int_0^\infty \frac{\sin(px)}{p} \exp(-mx)(\mu x)^D dx \; &= -\Gamma(a+1) \frac{\cos(\pi a)}{2} (p^2 + m^2)^{-1-a/2} \mu^a \frac{1}{\sqrt{-p^2}} \times \\ &\times \left[(\sqrt{-p^2} + m) \left(\frac{\sqrt{-p^2} - m}{\sqrt{-p^2} + m} \right)^{-a/2} + (\sqrt{-p^2} - m) \times \right. \\ &\left. \times \left(\frac{\sqrt{-p^2} + m}{\sqrt{-p^2} - m} \right)^{a/2} \right] \\ &\sim (\sqrt{-p^2} - m)^{-1-a} \text{ near } p^2 = -m^2, a = \frac{e^2}{8\pi m} . \end{aligned} \tag{2.41}$$

Formally fourier transform of $\sigma(p)$ can be written by double fourier transform

$$\sigma(p) = \int \frac{d^3 q}{(2\pi)^3} F.T. \left(\frac{\exp(-mx)}{x} (\mu x)^D \right) (q) \times F.T. (\mu x)^{-Cx} (p - q) , \tag{2.42}$$

but we do not discuss details of it in this paper. We have seen that the fourier transformation of the propagator is very difficult. It is interesting to study the phase structure of the model. First we estimate renormalization constant by the following sum rule [10, 11],

$$Z^{-1} = \int \sigma(\omega) d\omega , \tag{2.43}$$

$$Z^{-1} \int \frac{\sin(px)}{p} \frac{\exp(-m_0 x)}{4\pi x} dx = \int \sigma(\omega) d\omega \frac{\sin(px)}{p} \frac{\exp(-\omega x)}{4\pi x} dx , \tag{2.44}$$

$$\frac{Z^{-1}}{p^2 - m_0^2} = \int \frac{\sigma(\omega) d\omega}{p^2 - \omega^2 + i\epsilon} , \tag{2.45}$$

$$\lim_{p \to \infty} F(p) = \lim_{p \to \infty} \int_0^\infty \frac{\sin(px)}{p} F(x) dx = \lim_{p \to \infty} \frac{1}{p^2} F(0) , \tag{2.46}$$

we can evaluate this quantity by direct substitution of $\sigma(x)$ into the above equations and take the limit

$$\begin{aligned} Z^{-1} &= \exp(A) \lim_{x \to 0_+} \exp(-mx)(\mu x)^{-Cx+D-1} \\ &= \begin{bmatrix} 0 & (1 < D) \\ \text{finite} & (1 = D) \\ \infty & (1 > D) \end{bmatrix} . \end{aligned} \tag{2.47}$$

We see if the coupling constant D is smaller than 1,renormalization constant becomes $Z = 0(Z^{-1} = \infty)$ and it shows a signal of confinement of charged particle.

Next we consider the gauge dependence of the propagator [12]–[14].When we write the photon pagator as

$$D_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + k_\mu k_\nu M(k)\,, \tag{2.48}$$

we seek the change of the propagator. Under the gauge transformation defined:

$$\begin{aligned}\phi(x) &\to \exp(ie\chi(x)), \phi^+(x) \to \phi^+(x)\exp(-ie\chi(x))\,,\\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\chi, \delta(\phi^+ \overleftrightarrow{\partial_\mu}\phi) = -\partial_\mu\chi(\phi^+\overleftrightarrow{\partial_\mu}\phi - 2ie\phi^+\phi A_\mu)\,,\end{aligned} \tag{2.49}$$

propagator changes

$$\begin{aligned}D_{\mu\nu} &\to i\left\langle T[A_\mu(x) - \partial^x_\mu\chi(x)][A_\nu(y) - \partial^y_\nu(y)]\right\rangle\\ &= D_{\mu\nu} + i\left\langle T\partial^x_\mu\chi(x)\partial^y_\nu(y)\right\rangle,\\ \delta D_{\mu\nu} &= \partial^x_\mu\partial^y_\nu M(x-y), M(x-y) = i\left\langle T\chi(x)\chi(y)\right\rangle\\ \Delta &\to -i\left\langle T\phi(x)\exp(ie\chi(x)\exp(-ie\chi(y)\phi^+(y)\right\rangle\\ &= \Delta\left\langle T\exp(ie\chi(x))\exp(-ie\chi(y))\right\rangle.\end{aligned} \tag{2.50}$$

Here we expand in $\delta\chi$

$$\begin{aligned}\left\langle T\exp(ie\delta\chi(x))\exp(-ie\delta\chi(y))\right\rangle &= -\frac{1}{2}e^2\left\langle\delta\chi^2(x) + \delta\chi^2(y) - 2T\delta\chi(x)\delta\chi(y)\right\rangle,\\ \delta\Delta &= ie^2\Delta^{(0)}(x-y)[M(0) - M(x-y)]\,,\\ i[M(0) - M(x-y)] &= -\left\langle T\delta\chi(x)\delta\chi(y)\right\rangle.\end{aligned} \tag{2.51}$$

Usually M is identified as the gauge fixing term

$$\begin{aligned}M(k) &= -\frac{d}{k^4}\,,\\ ie^2M(x) &= -ie^2 d\int\frac{d^3k}{(2\pi)^3}\frac{\exp(ik\cdot x) - 1}{k^4}\\ &= -\frac{e^2d}{8\pi}x\,.\end{aligned} \tag{2.52}$$

We get the propagator (2.50) in covariant d gauge for three dimension

$$\begin{aligned}\Delta(d,x) &= \exp(-\frac{e^2d}{8\pi}x)\Delta(0,x)\\ &= \exp(-\frac{e^2d}{8\pi}x)\sigma(0,x)\,.\end{aligned} \tag{2.53}$$

Since $\sigma(x)$ is a full propagator it obeys the Landau-Kharatonikov transformation in our approximation

$$\sigma(d,x) = \frac{\exp(-(m^* + de^2/8\pi)x)}{x}\exp(A)(\mu x)^{-Cx+D}\,. \tag{2.54}$$

It is not clear that the point discussed by Delbourgo,Waites [6], spectral function in momentum space is gauge invariant for scalar which we do not discuss here. We keep the Landau gauge and not take the covariant d gauge to avoid longitudinal photon. Here we mention the radiative correction of mass. In the Landau gauge our approximation changes the mass m to $m^* = m + e^2(\gamma - 2)/(8\pi)$. The constant γ appeared after the expansion of $\mathrm{Ei}(1, \mu x)$ in μ. In ref. [5], gauge dependent self-energy is estimated at $O(e^2)$

$$\Sigma(m) = d\frac{e^2}{4\pi}\,, \tag{2.55}$$

and the physical mass for fermion defined in the Landau gauge

$$m_{phy} = m + \Sigma(m)\,. \tag{2.56}$$

If we estimate one-loop self-energy in the Landau gauge

$$\Sigma(m) = \frac{e^2}{2\pi}\,, \tag{2.57}$$

which is two-times larger than $e^2/(4\pi)$ in m^* since we used $D_+(x)$ instead of $D(x)$. We see the same effects occured for scalar with vertex correction. We call $m^* = m + e^2(\gamma - 2)/8\pi$ the renormalized mass in our approximation. At first sight, mass shift seems to be peculiar but infrared behaviour is governed by one-particle intermediate state.

3. Spinor QED

In this section we study the spectral function for massive fermion. Similar to scalar case propagator and the spectral function are defined

$$\begin{aligned} S(p) &= \int d^3x \exp(ip\cdot x) \left\langle \Omega | T(\psi(x)\overline{\psi}(0)) | \Omega \right\rangle \\ &= (\int_m^\infty + \int_{-\infty}^{-m}) \frac{\rho(\omega) d\omega}{p\cdot\gamma - \omega + i\epsilon}\,, \\ \rho(p^2) &= (2\pi)^2 \sum_N \delta^{(3)}(p - p_N) \left\langle \Omega | \psi | N \right\rangle \left\langle N | \overline{\psi} | \Omega \right\rangle . \end{aligned} \tag{3.1}$$

Matrix element is

$$\begin{aligned} T_n &= \langle \Omega | \psi | r; k_1, \ldots, k_n \rangle\,, \\ T_n^\mu &= -\int d^3x \exp(ik_n\cdot x) \langle \Omega | T\psi j^\mu | r; k_1, \ldots, k_{n-1} \rangle\,, \end{aligned} \tag{3.2}$$

provided

$$\Box_x T\psi A_\mu(x) = T\psi \Box_x A_\mu(x) = T\psi(-j_\mu(x) + \partial_\mu^x(\partial\cdot A(x)))\,. \tag{3.3}$$

In the similar way to the scalar case T_n satisfies Ward-Identity:

$$\begin{aligned} \partial_\mu^x T(\psi j_\mu(x)) &= -e\psi(x)\,, \\ \partial_\mu^x T(\overline{\psi} j_\mu(x)) &= e\overline{\psi}(x)\,, \\ k_{n\mu} T_n^\mu(r, k_1, k_2, \ldots, k_n) &= eT_{n-1}(r, k_1, k_2, \ldots, k_{n-1}), r^2 = m^2\,. \end{aligned} \tag{3.4}$$

One photon matrix element by LSZ is

$$
\begin{aligned}
T_1 &= \left\langle in|T(\psi_{in}(x), ie\int d^3y\overline{\psi}_{in}(y)\gamma_\mu\psi_{in}(y)A^\mu_{in}(y))|r;k\ in\right\rangle \\
&= ie\int d^3yd^3zS_F(x-y)\gamma_\mu\delta^{(3)}(y-z)\exp(i(k\cdot y+r\cdot z))\epsilon^\mu(k,\lambda)U(r,s) \\
&= -ie\frac{(r+k)\cdot\gamma+m}{((r+k)^2-m^2)}\gamma_\mu\epsilon^\mu(k,\lambda)\exp(i((k+r)\cdot x))U(r,s)\,,
\end{aligned}
\tag{3.5}
$$

where $U(r,s)$ is a two-component free particle spinor with positive energy:

$$
\sum_S U(r,s)\overline{U}(r,s) = \frac{\gamma\cdot r+m}{2m}\,. \tag{3.6}
$$

In this case the function F becomes

$$
\begin{aligned}
F &= e^2\int\frac{d^3k}{(2\pi)^3}\exp(ik\cdot x)\theta(k^0)\delta(k^2)\times \\
&\quad\times\ \mathrm{tr}\left[\frac{(r+k)\cdot\gamma+m}{((r+k)^2-m^2)}\gamma_\mu\frac{\gamma\cdot r+m}{2m}\gamma_\nu\frac{(r+k)\cdot\gamma+m}{((r+k)^2-m^2)}\Pi^{\mu\nu}\right] \\
&= -e^2\int\frac{d^3k}{(2\pi)^3}\exp(ik\cdot x)\theta(k^0)\times \\
&\quad\times\left[\delta(k^2-\mu^2)\left(\frac{m^2}{(r\cdot k)^2}+\frac{1}{(r\cdot k)}\right)+(d-1)\frac{\partial\delta(k^2-\mu^2)}{\partial k^2}\right],
\end{aligned}
\tag{3.7}
$$

$$
F = \frac{e^2m^2}{8\pi r^2}\left(-\frac{1}{\mu}+x(1-\gamma)-x\ln(\mu x)\right)+\frac{e^2}{8\pi r}(\ln(\mu x)+\gamma)+\frac{e^2}{8\pi}(d-1)\left(\frac{1}{\mu}-x\right). \tag{3.8}
$$

Thus the lowest order spectral function is exactly the same with that in the scalar case except for the normalization factor in the phase space integral:

$$
\begin{aligned}
\rho(p^2) &= \int\frac{d^3x}{(2\pi)^3}\exp(ip\cdot x)\rho(x) \\
&= \int\frac{d^3x}{(2\pi)^3}\exp(ip\cdot x)\int d^2r\frac{m}{r_0}\exp(ir\cdot x)\exp(F)\,.
\end{aligned}
\tag{3.9}
$$

After angular integral we get

$$
\rho(p) = \frac{m}{4\pi p}\exp(A)\int_0^\infty dx\sin(px)\exp(-(B+m)x)(\mu x)^{-Cx+D}\,. \tag{3.10}
$$

Therefore the structure is the same with scalar case as we mentioned in the last section. Consequently there is no infrared divergences at $p^2=m^2$.

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For the renormalization constants we apply the same argument as for the scalar,full propagator is expressed in the dipersion integral

$$
\begin{aligned}
S_F(x) &= -\int \rho(\omega) d\omega (i\gamma\cdot\partial + \omega)\frac{\exp(-\omega x)}{4\pi x} = -(i\gamma\cdot\partial\rho(x) + \rho(x))\,, \\
\frac{Z_2^{-1}}{\gamma\cdot p - m_0} &= \int \frac{\rho(\omega)d\omega}{\gamma\cdot p - \omega + i\epsilon} \\
Z_2^{-1} &= \int \rho(\omega)d\omega = \exp(A)\lim_{x\to 0_+}(\mu x)^{-Cx+D-1}\,, \\
\frac{m_0}{m} Z_2^{-1} &= \int \omega\rho(\omega)d\omega = \exp(A)\lim_{x\to 0_+}(\mu x)^{-Cx+D-1} \\
&= \exp(A)\mu^D \lim_{x\to 0_+}(x^{-Cx+D-1}) = \begin{bmatrix} 0 & (1<D) \\ \text{finite} & (1=D) \\ \infty & (1>D) \end{bmatrix}.
\end{aligned} \tag{3.11}
$$

Therefore there is a confining phase; $Z_2 = 0(Z_2^{-1} = \infty)$ of charged particle for weak coupling constant. Order parameter for the vacuum expectation value of pair condensate is given

$$
\begin{aligned}
\langle\overline{\psi}\psi\rangle &= -i\,\mathrm{tr}\,S_F(x) = -2m\mu^D \lim_{x\to 0_+}(\exp(-mx)(x)^{-Cx+D-1}) \\
&= \begin{bmatrix} 0 & (1<D) \\ \text{finite} & (1=D) \\ \infty & (1>D) \end{bmatrix}.
\end{aligned} \tag{3.12}
$$

Here we notice that there is a critical coupling constant $D_{cr} = 1(e^2/(8\pi m) = 1)$.

4. Summary

We have seen how the Ward-Identity for soft photon may be applied to three-dimensional electro-dynamics to extract full propagator in position space and the infrared behaviour of the propagator in momentum space. Neglecting unconventional terms in the elecrto-magnetic current for scalar,spectral functions for scalar and spinor coincide each other. Since the theory is super renormalizable the mass are corrected to add some finite radiative correction. Renormalization constant vanshes for both scalar and spinor cases and there is a confining phase for weak coupling. This picture is consistent with perturbative logarithmic infrared divergences at $p^2 = m^2$. If the coupling constant is smaller than D_{cr} ($D = e^2/8\pi m \leq 1$),vacuum expectation value $\langle\overline{\psi}\psi\rangle$ becomes finite. In our lowest order spectral function there remains linear infrared divergences that was regularized by photon mass. $O(e^2)$ correction to the bare vertex and the external line and the problems of cancellation of infrared divergences are now in progress [17].

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A. Evaluation of the one-photon matrix-element

To evaluate F we use the function $D_+(x)$ in three dimension:

$$\begin{aligned} D_+(x) &= \frac{1}{(2\pi)^2 i}\int \exp(ik\cdot x)\theta(k^0)\delta(k^2-\mu^2)d^3k \\ &= \frac{1}{(2\pi)^2 i}\int_0^\infty J_0(kx)\frac{\pi k dk}{2\sqrt{k^2+\mu^2}} = \frac{\exp(-\mu x)}{8\pi i x}\,, \end{aligned} \tag{A.1}$$

with the following parameter trick as in ref. [9]

$$\begin{aligned} \lim_{\epsilon\to 0_+}\int_0^\infty d\alpha \exp(i(k+i\epsilon)\cdot(x+\alpha r)) &= \frac{\exp(ik\cdot x)}{k\cdot r}\,, \\ \lim_{\epsilon\to 0_+}\int_0^\infty \alpha d\alpha \exp(i(k+i\epsilon)\cdot(x+\alpha r)) &= \frac{\exp(ik\cdot x)}{(k\cdot r)^2}\,. \end{aligned} \tag{A.2}$$

The function F is written in terms of the parameter integrals

$$\begin{aligned} F &= ie^2m^2\int_0^\infty \alpha d\alpha D_+(x+\alpha r,\mu) - e^2\int_0^\infty d\alpha D_+(x+\alpha r,\mu) - ie^2(d-1)\frac{\partial}{\partial\mu^2}D_+(x,\mu) \\ &= \frac{e^2m^2}{8\pi r^2}\left(-\frac{\exp(-\mu x)}{\mu} + x\,\mathrm{Ei}(1,\mu x)\right) - \frac{e^2}{8\pi r}\mathrm{Ei}(1,\mu x) + (d-1)\frac{e^2}{8\pi\mu}\exp(-\mu x)\,, \end{aligned} \tag{A.3}$$

where the function $\mathrm{Ei}(n,x)$ is defined

$$\mathrm{Ei}(n,x) = \int_1^\infty \frac{\exp(-xt)}{t^n}dt\,. \tag{A.4}$$

It is understood that all terms which vanish with $\mu\to 0$ are ignored.

$$\begin{aligned} \mathrm{Ei}(1,\mu x) &= -\gamma - \ln(\mu x) + O(\mu x)\,, \\ F_1 &= \frac{e^2m^2}{8\pi r^2}\left(-\frac{1}{\mu} + x(1-\ln(\mu x)-\gamma)\right) + O(\mu)\,, \\ F_2 &= \frac{e^2}{8\pi r}(\ln(\mu x)+\gamma) + O(\mu)\,, \\ F_g &= \frac{e^2}{8\pi}\left(\frac{1}{\mu} - x\right)(d-1) + O(\mu)\,. \end{aligned} \tag{A.5}$$

Here γ is Euler's constant. Using the integrals

$$\begin{aligned} \int d^3x \exp(ip\cdot x)\int d^3r\delta(r^2-m^2)\exp(ir\cdot x)f(r) &= f(m)\,, \\ \int \frac{d^3x}{(2\pi)^3}\exp(ip\cdot x)\int d^3r\delta(r^2-m^2) &= \frac{1}{m^2+p^2}\,, \end{aligned} \tag{A.6}$$

we get F in position space

$$F = \frac{e^2}{8\pi\mu}(d-2) + \frac{\gamma e^2}{8\pi r} + \frac{e^2}{8\pi r}\ln(\mu x) - \frac{e^2}{8\pi}x\ln(\mu x) - \frac{e^2}{8\pi}x(d+\gamma-2)\,. \tag{A.7}$$

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After integration over r

$$\int_0^\pi \exp(ir\cdot x\cos(\theta))d\theta = \pi J_0(rx)\,,$$
$$\int_0^\infty dr\frac{\pi r J_0(rx)}{\sqrt{r^2+m^2}} = \frac{\exp(-mx)}{x}\,, \tag{A.8}$$

we obtain the full propagator

$$\sigma(x) = \exp(A)\frac{\exp(-(m+B)x)}{4\pi x}(\mu x)^{-Cx+D}\,,$$
$$A = \frac{e^2}{8\pi\mu}(d-2)+\frac{\gamma e^2}{8\pi m}\,, \qquad B = \frac{e^2}{8\pi}(d+\gamma-2)\,, \qquad C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}\,. \tag{A.9}$$

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